Surface waves due to blasts on and above liquids

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The problem of surface waves due to the interaction of a blast-generated shock wave with an ideal incompressible heavy (or light) fluid of infinite depth has been investigated in both two and three dimensions. The wave integrals have been evaluated exactly for arbitrary as well as special pressure distributions on the fluid surface. Asymptotic values of the surface wave elevation have been obtained for large values of time at a large distance from the seat of the applied pressure. Certain peculiarities of the motion are discussed.

1. Introduction

The problem of surface waves caused by the interaction of a blast-generated shock wave with an ideal incompressible fluid has been investigated by Rumiantsev (1960) and Kisler (1960). In view of the large difference between the densities of the gas and the fluid, it is assumed that the fluid displacements are too small to affect the motion of the gas, which is supposed known. This leads us to a problem of infinitesimal wave motion due to a known unsteady pressure distribution over a time-varying area on the free surface of the fluid.

The work of Rumiantsev is concerned with finding a number of similarity solutions of the above problem for a weightless fluid of infinite depth in both two and three dimensions. Kisler, on the other hand, formulates the general three-dimensional gravity wave problem for shallow liquids, and then discusses the deep-water waves due to a special pressure distribution. The wave integral in the last case is made to depend on some other integrals which remain unevaluated in exact terms. Finally, the surface displacement of a weightless fluid is derived in this particular case.

In this paper, we first formulate the problem stated before for the case of a deep heavy liquid in two dimensions, and evaluate the resulting wave integral for arbitrary as well as special pressure distributions. An asymptotic expression for the wave elevation is given for large values of time at a large distance from the pressure zone. For a weightless fluid, a solution of the problem more general than Rumiantsev's is proposed. This two-dimensional problem has not been examined by the previous investigators. In the three-dimensional case, the deep-water gravity wave problem is discussed by a method different from Kisler's. The wave integral is analysed for arbitrary pressure distributions, and its exact evaluation, hitherto unknown, is given together with integrals suitable for numerical calculation. The general solution is illustrated for plausible laws of pressure variations, and incidentally, the exact solution of Kisler's special problem is also given. An asymptotic expression similar to the one in the two-dimensional case is obtained for the wave elevation. Finally, we derive similar expressions for a weightless fluid and discuss some points of interest.

2. The two-dimensional problem

We consider a semi-infinite ocean of a heavy homogeneous ideal liquid which is initially at rest. Its undisturbed horizontal free surface forms the (x, y)-plane, while the z-axis is taken vertically upwards.

Let us suppose that surface waves are excited in the ocean by a cylindrical pressure region propagating outwards from O, such as would be caused by a strong cylindrical blast on the liquid surface. The pressure function $p_0(x,t)$ is taken, for $t > 0$, as

$$\left.\begin{aligned} p_0(x,t) &= f(x,t) \quad (|x| < x_0(t)) \\ &= 0 \qquad\quad (|x| > x_0(t)). \end{aligned}\right\} \tag{1}$$

Here the function $f(x,t)$ is assumed to possess a Fourier transform in x. As the motion starts from rest, there exists a velocity potential $\phi(x,z;t)$ which satisfies the equation

$$\nabla^2 \phi = 0 \quad (z < 0, \quad t > 0). \tag{2}$$

On the assumption of small oscillations in which squares of the velocities are negligible, the pressure equation gives

$$p/\rho = \partial\phi/\partial t - gz,$$

where p denotes the pressure at (x,z) at time t, ρ the density of the liquid, and g the acceleration due to gravity. If $\zeta(x,t)$ denotes the surface elevation above the undisturbed surface, then we get

$$g\rho\zeta = -p_0(x,t) + \rho(\partial\phi/\partial t)_{z=0}. \tag{3}$$

The kinematical surface condition gives

$$\partial\zeta/\partial t = -(\partial\phi/\partial z)_{z=0} \quad (t > 0). \tag{4}$$

Eliminating ζ between (3) and (4), we get

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = \frac{1}{\rho}\frac{\partial p_0}{\partial t} \quad (z = 0, t > 0). \tag{5}$$

The condition at infinity is

$$\phi \to 0 \quad \text{for} \quad z \to -\infty \quad (t > 0). \tag{6}$$

The initial conditions are

$$\phi(x,0;0) = \phi_t(x,0;0) = 0. \tag{7}$$

The problem is to find a solution $\phi(x,z;t)$ of (2) which satisfies the equations (5) and (7) together with the condition (6). The surface elevation ζ is then found by (3).

Solution

Assuming a Fourier transform for the velocity potential function $\phi(x,z;t)$, namely

$$\bar{\phi}(k,z;t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\phi(x,z;t)\,e^{ikx}dx, \tag{8}$$

we obtain from (2) the equation

$$\partial^2\bar{\phi}/\partial z^2 - k^2\bar{\phi} = 0.$$

The solution of this equation which satisfies (6) is

$$\bar{\phi} = A(k,t)\,e^{|k|z}. \tag{9}$$

The transformation of (5) with the help of (9) gives the equation for A,

$$\frac{\partial^2 A}{\partial t^2} + g|k|\,A = \frac{1}{\rho\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{\partial p_0(x,t)}{\partial t}\,e^{ikx}dx$$
$$= \frac{1}{\rho\sqrt{(2\pi)}}\frac{\partial}{\partial t}\int_{-x_0(t)}^{x_0(t)} f(x,t)\,e^{ikx}dx.$$

The real solution of this equation is

$$A = A_0(k)\cos\{\sigma t - \sigma_0(k)\} + \frac{1}{\rho\sigma\sqrt{(2\pi)}}\int_0^t \sin\sigma(t-s)\,ds\,\frac{\partial}{\partial s}\int_{-x_0(s)}^{x_0(s)} f(x,s)\,e^{ikx}dx,$$

where

$$\sigma^2 = g|k|. \tag{10}$$

The initial conditions (7) give $A_0 = 0$. Consequently we get, after an integration by parts,

$$A = \frac{1}{\rho\sqrt{(2\pi)}}\int_0^t \cos\sigma(t-s)\,ds\int_{-x_0(s)}^{x_0(s)} f(x,s)\,e^{ikx}dx. \tag{11}$$

Inverting (8) by the Fourier inversion theorem, using (9) and (11), we find that the velocity potential is

$$\phi = (2\pi\rho)^{-1}\int_{-\infty}^{\infty} e^{|k|z-ikx}dk\int_0^t \cos\sigma(t-s)\,ds\int_{-x_0(s)}^{x_0(s)} f(\alpha,s)\,e^{ik\alpha}d\alpha. \tag{12}$$

Equation (3) now gives

$$-2\pi g\rho\zeta = \lim_{z\to 0-}\int_{-\infty}^{\infty} e^{|k|z-ikx}dk\int_0^t \sigma\sin\sigma(t-s)\,ds\int_{-x_0(s)}^{x_0(s)} f(\alpha,s)\,e^{ik\alpha}d\alpha. \tag{13}$$

Evaluation of the wave integral

Assuming that the interchange of the orders of k- and s-integrations in (13) is permissible, we have from (13) that

$$-\pi g\rho\zeta = \int_0^t ds\lim_{z\to 0-}\int_0^{\infty} dk\int_{-x_0(s)}^{x_0(s)} \sigma\sin\sigma(t-s)\,e^{kz}\cos k(x-\alpha)\,f(\alpha,s)\,d\alpha.$$

For values of x outside the domain of applied pressure, it is generally permissible to interchange the orders of the k- and α-integrals. This consideration leads to

$$-\pi g\rho\zeta = \int_0^t ds\int_{-x_0(s)}^{x_0(s)} f(\alpha,s)\,d\alpha\lim_{z\to 0-}\int_0^{\infty}\sigma\sin\sigma(t-s)\cos k(x-\alpha)\,e^{kz}dk. \tag{14}$$

Expanding $\sin\sigma(t-s)$ in powers of $\sigma(t-s)$ and integrating the series term by term, which is permissible, we obtain the value of the k-integral of (14) as

$$\sum_{n=0}^{\infty}(-1)^n \frac{g^{n+1}(t-s)^{2n+1}}{(2n+1)!}\int_0^{\infty} e^{kz}\,k^{n+1}\cos k(x-\alpha)\,dk$$
$$= \sum_{n=0}^{\infty}(-1)^n \frac{g^{n+1}(t-s)^{2n+1}}{(2n+1)!}\,\frac{(n+1)!\cos\{(n+2)\tan^{-1}[(\alpha-x)/z]\}}{[z^2+(x-\alpha)^2]^{(n+2)/2}}.$$

Therefore

$$\lim_{z\to 0-}(\text{the } k\text{-integral of (14)}) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{g^{2n+1}(t-s)^{4n+1}(2n+1)!}{(4n+1)(x-\alpha)^{2n+2}}, \quad |x| > x_0(t). \tag{15}$$

The last series in (15) can be summed in terms of tabulated functions. For, on simplifying the factorials in it by the multiplication theorem for Gamma functions, we find this series to be equal to

$$-\frac{g(t-s)}{(x-\alpha)^2}\frac{d}{d\omega}\left[\omega\frac{d}{d\omega}\{\omega S'\}\right] = -\chi, \quad \text{say,}$$

where $$\omega = g(t-s)^2/2(x-\alpha),$$

and
$$S' = {}_2F_3(1,\tfrac{1}{2};\tfrac{3}{2},\tfrac{3}{4},\tfrac{5}{4};-\tfrac{1}{16}\omega^2)$$
$$= {}_1F_1(\tfrac{1}{2};\tfrac{3}{2};\tfrac{1}{2}i\omega)\,{}_1F_1(\tfrac{1}{2};\tfrac{3}{2};-\tfrac{1}{2}i\omega)$$
$$= \pi\omega^{-1}[\{C(\tfrac{1}{2}\omega)\}^2+\{S(\tfrac{1}{2}\omega)\}^2]$$

(Erdélyi 1953, 4.3(5), 6.9.2(29, 30)). Here ${}_pF_q$ denotes a generalized hypergeometric series, and $C(x)$ and $S(x)$ are Fresnel's integrals defined by

$$[C(x), S(x)] = (2\pi)^{-\frac{1}{2}}\int_0^x [\cos t, \sin t]\,t^{-\frac{1}{2}}\,dt.$$

Thus, $$\chi = \{g(t-s)/2(x-\alpha)^2\}\,[\pi^{\frac{1}{2}}\omega^{-\frac{1}{2}}\{\cos\tfrac{1}{2}\omega C(\tfrac{1}{2}\omega)+\sin\tfrac{1}{2}\omega S(\tfrac{1}{2}\omega)\} - \pi^{\frac{1}{2}}\omega^{\frac{1}{2}}\{\sin\tfrac{1}{2}\omega C(\tfrac{1}{2}\omega)-\cos\tfrac{1}{2}\omega S(\tfrac{1}{2}\omega)\}+1]. \tag{16}$$

We have now by (14)

$$\pi g\rho\zeta = \int_0^t ds\int_{-x_0(s)}^{x_0(s)} \chi f(\alpha,s)\,d\alpha, \; |x| > x_0(t), \tag{17}$$

where χ is defined either by the series (15) or by its closed-form expression (16). It should be noticed that the function χ does not depend on the actual form of $f(\alpha, s)$ and that it can be conveniently calculated by using (15) or (16).

Special pressure distributions

Specific hypotheses about the pressure function $f(x,t)$ are needed to complete the integrations on the right-hand side of (17).

(i) Let
$$f(x,t) = f_m(t)\,|x|^m \quad (m > -1), \tag{18}$$

where $f_m(t)$ is a known function of t. We insert the series for χ in (17), and integrate this series with respect to α term by term, which is permissible for $|x| > x_0(t)$.

A well-known result on Mellin transforms (Erdélyi 1954, 6.2(20)) now leads to the expression

$$\pi\rho(1+m)\,\zeta = \int_0^t f_m(s)\,ds \sum_{n=0}^{\infty}(-1)^n g^{2n}(t-s)^{4n+1}x^{-(2n+2)}\{x_0(s)\}^{1+m}$$
$$\times\frac{(2n+1)!}{(4n+1)!}\left[{}_2F_1\left(2n+2,1+m;2+m;\frac{x_0(s)}{x}\right)\right.$$
$$\left.+\,{}_2F_1\left(2n+2,1+m;2+m;-\frac{x_0(s)}{x}\right)\right],\quad |x|>x_0(t). \tag{19}$$

From this, one may construct more general solutions by superposition, especially when the pressure function is prescribed in the form

$$f(x,t) = \sum_{m=0}^{\infty} f_m(t)\,|x|^m.$$

(ii) In the particular case

$$\left.\begin{aligned} f(x,t) &= D|x|^m t^p \quad (m > -1, p+m+2 > 0);\\ x_0(t) &= vt, \end{aligned}\right\} \tag{20}$$

where D and v are constants, we find from (19), by using a known integral (Erdélyi 1954, 6.2),

$$\pi\rho(1+m)\,\zeta = Dt^{p+2}(vt)^{1+m}x^{-2}\sum_{n=0}^{\infty}(-1)^n(gt^2/x)^{2n}\frac{(2n+1)!\,\Gamma(p+m+2)}{\Gamma(4n+p+m+4)}$$
$$\times[{}_3F_2(2n+2,1+m,p+m+2;2+m,4n+p+m+4;vt/x)$$
$$+\text{a }{}_3F_2\text{ function obtained by replacing } vt/x \text{ by } -vt/x]\quad (|x|>vt). \tag{21}$$

Approximations

(i) *Series approximation.* For small values of $gt^2/|x-vt|$ or gt^2/x, the function χ in (17) may be replaced by the first two terms of its series representation (15), the result giving an approximate expression for ζ. In the particular problems (18) and (20), the first two terms of the n-series of (19) and (21) give the required approximate values of ζ. For instance, when

$$f(x,t) = D,\quad x_0(t) = vt,$$

equation (21) gives approximately

$$\pi\rho\zeta = Dxv^{-2}[2x'+(1-x')\ln(1-x')-(1+x')\ln(1+x')$$
$$-\tfrac{1}{60}g^2x^2v^{-4}\{20x'+\tfrac{110}{3}x'^3-x'^5+10(1-x')^3\ln(1-x')$$
$$-10(1+x')^3\ln(1+x')\}], \tag{22}$$

where
$$x' = vt/x \quad\text{and}\quad |x'|<1. \tag{23}$$

(ii) *Asymptotic representation of the surface displacement.* Let us take

$$f(x,t) = D(t+t_1)^{-n}\quad (n>0,\ x_0(t)=vt), \tag{24}$$

where D and v are constants. As this pressure distribution is symmetrical about the origin, it would be sufficient to consider values of ζ on the positive side of the x-axis only. By (13), we have

$$\pi D^{-1}g^{\frac{1}{2}}\rho\zeta = \int_0^\infty k^{-\frac{1}{2}}\cos kx\,dk\int_0^t -2\sin\sigma(t-s)\sin(kvs)\,(t_1+s)^{-n}\,ds. \tag{25}$$

Let us introduce the functions

$$C(x,a) = \int_x^\infty t^{a-1}\cos t\,dt, \quad S(x,a) = \int_x^\infty t^{a-1}\sin t\,dt. \tag{26}$$

The value of the s-integral of (25) may now be written as

$$\Delta_{c_{1-n}}(k,v,t_1)\cos P(k,v) + \Delta_{s_{1-n}}(k,v,t_1)\sin P(k,v)$$
$$-\Delta_{c_{1-n}}(k,-v,t_1)\cos P(k,-v) - \operatorname{sgn}(\sigma-kv)\,\Delta_{s_{1-n}}(k,-v,t_1)\sin P(k,-v),$$

where

$$\Delta_{c_{1-n}}(k,-v,t_1) = |\sigma-kv|^{n-1}\left[C\{(t+t_1)\,|\sigma-kv|, 1-n\} - C\{t_1|\sigma-kv|, 1-n\}\right],$$
$$P(k,v) = \sigma(t+t_1)+kvt_1.$$

The resulting k-integral of (25) is next evaluated asymptotically by the application of the method of stationary phase under the conditions

$$x \gg vt > vt_1, \quad g(t+t_1)^2/4x \gg 1, \tag{27}$$

it being supposed that the ratio t_1/t is not too small. The first of these conditions signifies that we are calculating ζ at a large distance from the source while the second follows from Lamb's condition (Lamb 1932, §241) for the applicability of the stationary phase formula. In the actual calculations, the products $\cos kx \cos P(k, \pm v)$ and $\cos kx \sin P(k, \pm v)$ are expressed as sums of cosines and sines, and only those phase terms are retained which have a stationary point in $0 < k < \infty$. The other phase terms are neglected because their contributions to the integral, as is well known, are of the second order on the scale of (27). We have then

$$\pi^{\frac{1}{2}}D^{-1}g^{\frac{1}{2}}\rho\zeta \sim (x-vt_1)^{-\frac{1}{2}}\left[\Delta_{c_{1-n}}(k_0,v,t_1)\cos(T_v^2-\tfrac{1}{4}\pi) + \Delta_{s_{1-n}}(k_0,v,t_1)\sin(T_v^2-\tfrac{1}{4}\pi)\right]$$

$-$ an expression obtained by replacing v by $-v$, where

$$k_0(v) = g(t+t_1)^2/4(x-vt_1)^2, \quad T_v^2 = g(t+t_1)^2/4(x-vt_1),$$

We observe that

$$\begin{aligned}\Delta_{c_{1-n}}(k_0,v,t_1) = {} & [\tfrac{1}{4}g(t+t_1)(2x+vt-vt_1)(x-vt_1)^{-2}]^{n-1} \\ & \times [C\{\tfrac{1}{4}g(t+t_1)^2(2x+vt-vt_1)(x-vt_1)^{-2}, 1-n\} \\ & - C\{\tfrac{1}{4}gt_1(t+t_1)(2x+vt-vt_1)(x-vt_1)^{-2}, 1-n\}].\end{aligned}$$

The functions $C(x,a)$ and $S(x,a)$ are actually equal to Böhmer's integrals or are proportional to the sine and the cosine integrals ($a = 0$) (Erdélyi 1953, 9.8 and 9.10). By using their Taylor expansions about a point $x = x_1 > 0$, their asymptotic expansions for $x \gg 1$ and the order relations (27), we ultimately obtain the following asymptotic expression for ζ

$$\zeta \sim -4D\rho^{-1}\pi^{-\frac{1}{2}}g^{-\frac{3}{2}}x^{\frac{1}{2}}(t+t_1)^{-n-1}\sin[gvt(t+t_1)^2/4x^2]\cos[g(t+t_1)^2/4x+\tfrac{1}{4}\pi]. \tag{28}$$

For $t_1 = 0{\cdot}5$ sec, $g = 9{\cdot}8$ m/sec^2, $v = 0{\cdot}05$ m/sec, $x = 10$ m, $n = 1$, the variation of ζ with t is illustrated in figure 1.

(iii) *Case of a weightless fluid.* In the immediate area of the blast, the pressure effects predominate over gravity while outside this region gravity effects are more important. This aspect of the motion leads us to study the case of an incompressible weightless fluid both inside the expanding pressure zone and in the immediate neighbourhood outside it. Letting $g = 0$ in (12) and (13), we get

$$\phi(x,z;t) = (2\pi\rho)^{-1}\int_{-\infty}^{\infty} e^{|k|z-ikx}dk\int_0^t ds\int_{-x_0(s)}^{x_0(s)} f(\alpha,s)\,e^{ik\alpha}d\alpha, \tag{29}$$

$$-2\pi\rho\zeta = \lim_{z\to 0-}\int_{-\infty}^{\infty}|k|\,e^{|k|z-ikx}\,dk\int_0^t (t-s)\,ds\int_{-x_0(s)}^{x_0(s)} f(\alpha,s)\,e^{ik\alpha}d\alpha. \tag{30}$$

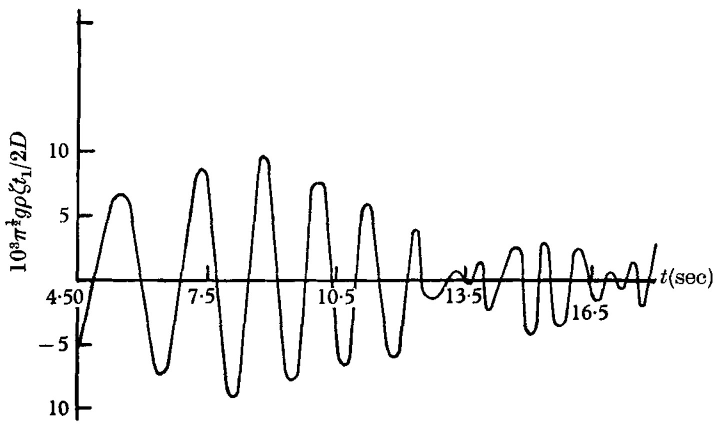

FIGURE 1

In general, these expressions may be reduced to

$$\phi = -z(\pi\rho)^{-1}\int_0^t ds\int_{-x_0(s)}^{x_0(s)}[z^2+(x-\alpha)^2]^{-1}f(\alpha,s)\,d\alpha, \tag{31}$$

$$\pi\rho\zeta = \int_0^t (t-s)\,ds\int_{-x_0(s)}^{x_0(s)}(x-\alpha)^{-2}f(\alpha,s)\,d\alpha \quad (|x| > x_0(t)). \tag{32}$$

For certain special pressure distributions, we obtain the following results.

(*a*) Let $\quad f(x,t) = D(t+t_1)^{-1}, \quad x_0(t) = vt \quad (D, v = \text{const.}).$ (33)

By (30) we have

$$\pi D^{-1}\rho\zeta = 2\int_0^\infty \cos kx[(t+t_1)\{(\mathrm{si}\,kvt_1-\mathrm{si}\,kv(t+t_1))\cos kvt_1 \\ -(\mathrm{ci}\,kvt_1-\mathrm{ci}\,kv(t+t_1))\sin kvt_1\}+(kv)^{-1}(1-\cos kvt)]\,dk$$

where si(z) and ci(z) denote the sine and the cosine integrals respectively. Using certain well-known results (Ryshik & Gradstein 1957, 4.234), we get

$$2\pi D^{-1}\rho\zeta = (t+t_1)\left[(x+vt_1)^{-1}\ln\left\{\frac{x(t+t_1)}{t_1(x-vt)}\right\}^2+(x-vt_1)^{-1} \\ \times\ln\left\{\frac{(x+vt)\,t_1}{x(t+t_1)}\right\}^2\right]+4v^{-1}\ln\frac{\sqrt{|v^2t^2-x^2|}}{|x|} \quad (x \neq 0, \pm vt, \pm vt_1). \tag{34a}$$

When $x = \pm vt_1$,

$$\pi D^{-1}\rho\zeta = -\frac{t}{vt_1} + \frac{t+t_1}{4vt_1}\ln\left(\frac{t+t_1}{t-t_1}\right)^2 + \frac{2}{v}\ln\frac{\sqrt{|t^2-t_1^2|}}{t_1}. \tag{34b}$$

Observing that $\lim_{x\to\pm vt}\zeta$ exists finitely for all time, we conclude that ζ has only an ordinary discontinuity at the boundary of the advancing pressure region. The function $(\zeta)_{x=\pm vt_1}$ has also an ordinary discontinuity at $t = t_1$.

(*b*) With
$$f(x,t) = (K/t)\,G(\xi),$$

where $\xi = x/x_0(t)$, $\eta = z/x_0(t)$, $x_0(t) = ct^{\frac{1}{2}}$, and K, c = const., the expression (31) leads to Rumiantsev's (1960) similarity solution of the corresponding problem.

(*c*) A more general solution of the type in (*b*) is easily constructed. Let us introduce the dimensionless variables

$$\xi = x/x_0(t), \quad \eta = z/x_0(t), \tag{35}$$

and assume a pressure function of the form

$$f(x,t) = \{x_0'(t)/x_0(t)\}\,KG(x/x_0(t)), \tag{36}$$

where K is a constant of the dimensions of a velocity potential. Let

$$Kw(z') \quad (z' = \xi + i\eta),$$

be the complex velocity potential. Then (31) gives

$$\frac{dw}{dz'} = \frac{i}{\pi\rho z'}\int_{-1}^{1}\frac{G(\alpha)}{z'-\alpha}\,d\alpha. \tag{37}$$

This solution is similar in form to that obtained by Rumiantsev for $x_0(t) = ct^{\frac{1}{2}}$.

In particular, if
$$G(\xi) = |\xi|^n \quad (n \geqslant 0) \tag{38}$$

we get
$$\frac{dw}{dz'} = \frac{2i}{\pi\rho(n+1)z'^2}\,{}_2F_1\left(1,\frac{n+1}{2};\frac{n+3}{2};\frac{1}{z'^2}\right), \quad \left|\arg\left(1-\frac{1}{z'^2}\right)\right| < \pi. \tag{39}$$

On the hypothesis (36), we have

$$(\zeta)_{y=0} = (\pi\rho x^2)^{-1}\int_0^t (t-s)\,x_0'(s)\,ds\int_{-1}^{1}\{1-\alpha x^{-1}x_0(s)\}^{-2}G(\alpha)\,d\alpha. \tag{40}$$

Together with (38), this gives

$$(\zeta)_{y=0} = \{\pi\rho x^2(n+1)\}^{-1}\int_0^t (t-s)\,x_0'(s)\,ds \times [{}_2F_1(2,n+1;n+2;x_0(s)/x) + {}_2F_1(2,n+1;n+2;\,-x_0(s)/x)] \quad (|x| > x_0(t)). \tag{41}$$

This value of the vertical displacement reduces to that obtained by Rumiantsev for the case $x_0(t) = ct^{\frac{1}{2}}$ and $n = 0$, c being a constant.

3. The three-dimensional problem

Let us suppose that surface waves are generated when the spherical shock wave due to a point blast in the gas interacts with the fluid surface. This gives rise to an expanding circular region of pressure on the free surface. With the cylindrical co-ordinates (r, θ, z) now replacing the Cartesian co-ordinates (x, y, z), the equation (1) to (3), (5) and (7) change into the following:

$$\text{for} \quad t > 0, \quad \left.\begin{aligned} p_0(r,t) &= f(r,t) \quad (r < r_0(t)) \\ &= 0 \qquad\quad (r > r_0(t)) \end{aligned}\right\} \tag{42}$$

(here the function $f(r,t)$ is assumed to possess a Hankel transform with respect to r);

$$\nabla^2\phi(r, z; t) = 0 \quad (z < 0, t > 0); \tag{43}$$

$$g\rho\zeta = -p_0(r,t) + \rho(\partial\phi/\partial t)_{z=0}; \tag{44}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = \frac{1}{\rho}\frac{\partial p_0}{\partial t} \quad (z = 0, t > 0); \tag{45}$$

$$\phi(r, 0; 0) = \phi_t(r, 0; 0) = 0, \tag{46}$$

while the conditions at infinity are the same as before.

Solution

We assume a solution of (43) of the form

$$\phi = \int_0^\infty A(k,t)\, e^{kz} J_0(kr)\, dk. \tag{47}$$

Substituting this in (45), we obtain by applying the Hankel inversion theorem the following differential equation for A:

$$\frac{\partial^2 A}{\partial t^2} + gkA = \frac{k}{\rho}\frac{\partial}{\partial t}\int_0^{r_0(t)} \alpha f(\alpha, t)\, J_0(k\alpha)\, d\alpha.$$

The real solution of this equation is

$$A = A_0(k)\cos(\sigma t + \epsilon_k) + \frac{k}{\rho\sigma}\int_0^t \sin\sigma(t-s)\frac{\partial}{\partial s}\int_0^{r_0(s)} \alpha f(\alpha, s)\, J_0(k\alpha)\, d\alpha\, ds, \tag{48}$$

where

$$\sigma^2 = gk. \tag{49}$$

Using (46), we get $A_0 = 0$. Equation (48) now gives, after an integration by parts,

$$A = k\rho^{-1}\int_0^t \cos\sigma(t-s)\, ds \int_0^{r_0(s)} \alpha f(\alpha, s)\, J_0(k\alpha)\, d\alpha.$$

The velocity potential is therefore

$$\phi = \rho^{-1}\int_0^\infty k e^{kz} J_0(kr)\, dk \int_0^t \cos\sigma(t-s)\, ds \int_0^{r_0(s)} \alpha f(\alpha, s)\, J_0(k\alpha)\, d\alpha. \tag{50}$$

The surface elevation is now found to be

$$\zeta = -(g\rho)^{-1}\lim_{z\to 0-}\int_0^\infty \sigma k J_0(kr)\, e^{kz} dk \int_0^t \sin\sigma(t-s)\, ds \int_0^{r_0(s)} \alpha f(\alpha, s)\, J_0(k\alpha)\, d\alpha, \tag{51}$$

on application of the Hankel inversion theorem.

Evaluation of the wave integral

We assume that the orders of k- and s-integrals in (51) can be interchanged. For values of r outside the region of applied pressure, it is generally permissible to invert the orders of k- and α-integrals.

Equation (51) now takes the form

$$g\rho\zeta = -\int_0^t ds \int_0^{r_0(s)} \alpha f(\alpha, s)\, d\alpha \lim_{z\to 0-} \int_0^\infty \sigma k e^{kz} \sin\sigma(t-s)\, J_0(kr)\, J_0(k\alpha)\, dk.$$

By Neumann's addition theorem for Bessel functions, we have

$$\pi J_0(kr)\, J_0(k\alpha) = \int_0^\pi J_0(k\lambda)\, d\theta,$$

where
$$\lambda = (r^2+\alpha^2-2r\alpha\cos\theta)^{\frac{1}{2}}. \tag{52}$$

We have now

$$\pi g\rho\zeta = -\int_0^t ds \int_0^{r_0(s)} \alpha f(\alpha, s)\, d\alpha \int_0^\pi d\theta \lim_{z\to 0-} \int_0^\infty \sigma k e^{kz} J_0(k\lambda) \sin\sigma(t-s)\, dk. \tag{53}$$

in which a permissible change of order of k- and θ-integrals has been effected. Expanding $\sin\sigma(t-s)$ in powers of $\sigma(t-s)$ and integrating the series with respect to k term by term (which is justifiable), we get

$$\text{the } k\text{-integral of (53)} = \sum_{n=0}^{\infty} (-1)^n \frac{g^{n+1}(t-s)^{2n+1}}{(2n+1)!} \int_0^\infty k^{n+2} e^{kz} J_0(k\lambda)\, dk.$$

By using Callandreau's integral representation (Whittaker & Watson 1952, p. 364) for Legendre polynomials $P_n(\cos\theta)$, we obtain

$$\int_0^\infty k^{n+2} e^{kz} J_0(k\lambda)\, dk = (n+2)!\,(z^2+\lambda^2)^{-\frac{1}{2}(n+3)} P_{n+2}[-z(z^2+\lambda^2)^{-\frac{1}{2}}].$$

Therefore

$$\lim_{z\to 0-} (\text{the } k\text{-integral of (53)}) = \frac{g(t-s)}{4\lambda^3} \sum_{n=0}^{\infty} (-1)^{n+1} \left\{\frac{g(t-s)^2}{2\lambda}\right\}^{2n} \frac{\{(2n+2)!\}^2}{(4n+1)!\,\{(n+1)!\}^2}. \tag{54}$$

After expressing the factorials in the above series in terms of simple similar factors by the multiplication theorem for Gamma functions, we apply to the result Schläfli and Schönholzer's series formula (Whittaker & Watson, 1952, p. 380) for products of Bessel functions. The right-hand side of (54) is then equal to

$$-(\pi/8\sqrt{2})\, g(t-s)\, \lambda^{-3}[(1-2\omega_1^4) J_{\frac{1}{4}} J_{-\frac{1}{4}} + 3\omega_1^2 (J_{-\frac{3}{4}} J_{-\frac{1}{4}} - J_{\frac{3}{4}} J_{\frac{1}{4}}) - 2\omega_1^4 J_{\frac{3}{4}} J_{-\frac{3}{4}}], \tag{55}$$

where
$$\tfrac{1}{2}\omega_1^2 = g(t-s)^2/8\lambda \tag{56}$$

is the common argument of all the Bessel functions written in (55). Writing

$$A_1 = g(t-s)^2/8(r+\alpha), \quad K_1^2 = 4r\alpha(r+\alpha)^{-2}, \quad \Delta \equiv \Delta(\theta, K_1^2) = (1-K_1^2\cos^2\theta)^{\frac{1}{2}}, \tag{57}$$

we have finally

$$4\sqrt{2}\rho\zeta = \int_0^t (t-s)\,ds \int_0^{r_0(s)} \alpha(r+\alpha)^{-3} f(\alpha,s)\,d\alpha \int_0^{\frac{1}{2}\pi} \Delta^{-3}\,d\theta$$
$$\times [(1-8A_1^2\Delta^{-2})J_{\frac{1}{4}}J_{-\frac{1}{4}} + 6A_1\Delta^{-1}(J_{-\frac{3}{4}}J_{-\frac{1}{4}} - J_{\frac{3}{4}}J_{\frac{1}{4}}) - 8A_1^2\Delta^{-2}J_{\frac{3}{4}}J_{-\frac{3}{4}}]$$
$$(r > r_0(t)), \quad (58)$$

where $A_1\Delta^{-1}$ is the common argument of all the Bessel functions written above. It may be shown that this result reduces to the expression obtained for ζ by Kisler (1960, equation (3.11)) for the particular case

$$f(r,t) = \sum_{n=0}^{\infty} r^{2n}\lambda_n(t).$$

The θ-integral may be conveniently evaluated by numerical integration as a function of the parameters A_1 and K_1^2; this function does not depend on the actual law of pressure variations.

Development of ζ in an infinite series

By (53) and (54), we have

$$4\pi\rho\zeta = r^{-3}\int_0^t (t-s)\,ds \int_0^{r_0(s)} \alpha f(\alpha,s)\,d\alpha \sum_{n=0}^{\infty} (-1)^n \left\{\frac{g(t-s)^2}{2r}\right\}^{2n}$$
$$\times \frac{[(2n+2)!]^2}{(4n+1)!\,[(n+1)!]^2}\int_0^{\pi}\left(1+\frac{\alpha^2}{r^2}-\frac{2\alpha}{r}\cos\theta\right)^{-n-\frac{3}{2}} d\theta \quad (r > r_0(t)),$$

in which the n-series is integrated term by term with respect to θ, which is clearly permissible for the range of values of r stated. Evaluating the θ-integral by means of a known result of the theory of the Gauss hypergeometric series ${}_2F_1(\alpha,\beta;\,\gamma;z)$ (Erdélyi 1953, 2.4(9)), we get

$$\pi\rho\zeta = 4r^{-3}\int_0^t (t-s)\,ds \int_0^{r_0(s)} \alpha f(\alpha,s)\,d\alpha$$
$$\times \sum_{n=0}^{\infty} (-1)^n \{2gr^{-1}(t-s)^2\}^{2n} \frac{\{\Gamma(n+\frac{3}{2})\}^2}{(4n+1)!}\, {}_2F_1\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}; 1; \frac{\alpha^2}{r^2}\right) \quad (r > r_0(t)),$$

on simplifying the factorials by the multiplication theorem for Gamma functions. The absolute convergence of the last series for all values of $gr^{-1}(t-s)^2$, $r > r_0(t)$ is easily established. For the relations (Erdélyi 1953, 2.8 (35, 38, 41)) between contiguous hypergeometric series give

$$\frac{{}_2F_1(a+1,a+1;1;z)}{{}_2F_1(a,a;1;z)} = \frac{a+z(3a-2)}{a(1-z)^2} + \frac{2z(a-1)^2}{a(1-z)^2}\frac{{}_2F_1(a,a;2;z)}{{}_2F_1(a,a;1;z)},$$

and the desired result follows by a ratio test. It is also clear that the series in question can be integrated with respect to α term by term for $0 \leqslant \alpha \leqslant r_0$. We have thus

$$\zeta = (4/\pi\rho)\,r^{-3}\int_0^t (t-s)\,ds \sum_{n=0}^{\infty} (-1)^n \{2gr^{-1}(t-s)^2\}^{2n} \frac{\{\Gamma(n+\frac{3}{2})\}^2}{(4n+1)!}$$
$$\times \int_0^{r_0(s)} \alpha f(\alpha,s)\, {}_2F_1\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}; 1; \frac{\alpha^2}{r^2}\right) d\alpha \quad (r > r_0(t)). \quad (59)$$

This result constitutes the exact series solution of the general problem, the functions $f(\alpha, s)$ and $r_0(s)$ being known from the solution of the corresponding gas-dynamical problem.

Particular cases

(i) Let
$$f(r,t) = \lambda_m(t) r^m \quad (m > -\tfrac{3}{2}). \tag{60}$$

By a well-known result on Mellin transforms (Erdélyi 1954, 6.2) we have

$$\int_0^{r_0(s)} \alpha^{m+1}\, {}_2F_1\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}; 1; \frac{\alpha^2}{r^2}\right) d\alpha = \tfrac{1}{2}B(1, \tfrac{1}{2}m+1)\{r_0(s)\}^{m+2}$$
$$\times\, {}_3F_2\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}, \tfrac{1}{2}m+1; 1, \tfrac{1}{2}m+2; \frac{r_0^2}{r^2}\right),$$

where B denotes the Beta function and ${}_pF_q$ the generalized hypergeometric series. Therefore

$$\zeta = (2/\pi\rho)\, r^{-3} B(1, \tfrac{1}{2}m+1) \int_0^t (t-s)\, \lambda_m(s) \{r_0(s)\}^{m+2}\, ds$$
$$\times \sum_{n=0}^{\infty} (-1)^n \{2gr^{-1}(t-s)^2\}^{2n} \frac{[\Gamma(n+\frac{3}{2})]^2}{(4n+1)!}\, {}_3F_2\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}, \tfrac{1}{2}m+1; 1, \tfrac{1}{2}m+2; \frac{r_0^2}{r^2}\right)$$
$$(r > r_0(t)). \tag{61}$$

For $m = 0$, the ${}_3F_2$ series of (61) reduces to a ${}_2F_1$ series.

(ii) The wave elevation due to a more general pressure distribution of the type

$$f(r,t) = \sum_{m=0}^{\infty} \lambda_m(t)\, r^m \tag{62}$$

is now easily constructed by the method of superposition. In particular, when only even powers of r are present in the series of (62), the result of superposition is the exact solution of Kisler's problem.

(iii) Let
$$\left.\begin{aligned} f(r,t) &= D r^m t^p \quad (m > -\tfrac{3}{2},\ p + \tfrac{1}{2}m + 2 > 0), \\ r_0(t) &= (kt)^{\frac{1}{2}} \quad (k = \text{const.}). \end{aligned}\right\} \tag{63}$$

Then, from (61),

$$\zeta = \frac{2\Gamma(p+\frac{1}{2}m+2)}{\pi(m+2)\, g\rho r^2} D t^p (kt)^{\frac{1}{2}m+1} \sum_{n=0}^{\infty} (-1)^n (2gr^{-1}t^2)^{2n+1} \frac{\{\Gamma(n+\frac{3}{2})\}^2}{\Gamma(4n+p+\frac{1}{2}m+4)}$$
$$\times\, {}_4F_3\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}, \tfrac{1}{2}m+1, p+\tfrac{1}{2}m+2; 1, \tfrac{1}{2}m+2, 4n+p+\tfrac{1}{2}m+4; \frac{kt}{r^2}\right)$$
$$(r > (kt)^{\frac{1}{2}}). \tag{64}$$

Approximations

(i) *Asymptotic representation of the wave elevation.* Let us take

$$f(r,t) = D(t+t_1)^{-n} \quad (n \geqslant 1; r_0(t) = vt), \tag{65}$$

where D and v are constants. By (51), we have

$$-(Dv)^{-1} g\rho\zeta = \int_0^{\infty} \sigma J_0(kr)\, dk \int_0^t s(t_1+s)^{-n} \sin\sigma(t-s)\, J_1(kvs)\, ds.$$

Replacing the Bessel functions by the corresponding finite integrals, and performing certain justifiable changes of the orders of integration, we get

$$(Dv)^{-1}\pi^2 g^{\frac{1}{2}}\rho\zeta = \int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta\int_0^{\frac{1}{2}\pi}d\phi\int_0^{\infty}2k^{\frac{1}{2}}\cos(kr\sin\phi)\,dk$$
$$\times\int_0^t(-2)\,s(t_1+s)^{-n}\sin\sigma(t-s)\sin(kvs\sin\theta)\,ds. \qquad (66)$$

The exact value of the s-integral of (66) is

$$[\Delta_{c_{2-n}}(k,v,t_1)-t_1\Delta_{c_{1-n}}(k,v,t_1)]\cos Q(k,v)$$
$$+[\Delta_{s_{2-n}}(k,v,t_1)-t_1\Delta_{s_{1-n}}(k,v,t_1)]\sin Q(k,v)$$
$$-[\Delta_{c_{2-n}}(k,-v,t_1)-t_1\Delta_{c_{1-n}}(k,-v,t_1)]\cos Q(k,-v)$$
$$-\operatorname{sgn}(\sigma-kv\sin\theta)\,[\Delta_{s_{2-n}}(k,-v,t_1)-t_1\Delta_{s_{1-n}}(k,-v,t_1)]\sin Q(k,-v),$$

where

$$\Delta_{c_a}(k,-v,t_1) = |\sigma-kv\sin\theta|^{-a}\{C[(t+t_1)|\sigma-kv\sin\theta|,a]-C[t_1|\sigma-kv\sin\theta|,a]\},$$
$$Q(k,v) = \sigma(t+t_1)+kvt_1\sin\theta;$$

the C and S functions used here are defined as in (26). It may be noted that the difference of the C functions in Δ_{c_a} reduces to a difference of sines when $a = 1$.

The resulting k-integral of (66) is next evaluated asymptotically by the application of the method of stationary phase under the conditions

$$r \gg vt > vt_1, \quad g(t+t_1)^2/4r \gg 1, \qquad (67)$$

it being supposed that the ratio t_1/t is not too small. In the actual calculations, the products $\cos(kr\sin\phi)\cos Q(k,\pm v)$ and $\cos(kr\sin\phi)\sin Q(k,\pm v)$ are expressed as sums of sines and cosines and only those phase terms are retained which have a stationary point $k = k_0$ in $0 < k < \infty$ for values of ϕ greater than $\sin^{-1}(vt_1\sin\theta/r)$. It is easily shown that such a stationary point for other values of ϕ would make the resulting ϕ-integral contribute a second-order term to ζ. After evaluating the k-integral of (66) by the stationary-phase formula, we determine the asymptotic value of the ϕ-integral so derived by the same method. We get

$$\sqrt{2\pi}\rho(Dv)^{-1}\zeta\sim\int_0^{\frac{1}{2}\pi}r^{-2}(t+t_1)\left(1-\frac{vt_1}{r}\sin\theta\right)^{-\frac{3}{2}}\sin\theta$$
$$\times[\{\Delta_{c_{2-n}}(k_0,v,t_1)-t_1\Delta_{c_{1-n}}(k_0,v,t_1)\}\cos L(v,\theta)$$
$$+\{\Delta_{s_{2-n}}(k_0,v,t_1)-t_1\Delta_{s_{1-n}}(k_0,v,t_1)\}\sin L(v,\theta)]\,d\theta$$
$$-\text{ an expression obtained by replacing } v \text{ by } -v, \qquad (68)$$

where $\quad k_0 = g(t+t_1)^2/4(r-vt_1\sin\theta)^2, \quad L(v,\theta) = g(t+t_1)^2/4(r-vt_1\sin\theta).$

We note that

$$\Delta_{c_{2-n}}(k_0,v,t_1) = \left[\frac{g(t+t_1)\{2r+v(t-t_1)\sin\theta\}}{4(r-vt_1\sin\theta)^2}\right]^{n-2}$$
$$\times\left(C\left[\frac{g(t+t_1)^2\{2r+v(t-t_1)\sin\theta\}}{4(r-vt_1\sin\theta)^2},2-n\right]\right.$$
$$\left.-C\left[\frac{gt_1(t+t_1)\{2r+v(t-t_1)\sin\theta\}}{4(r-vt_1\sin\theta)^2},2-n\right]\right).$$

The integrand on the right-hand side of (68) is simplified by using the asymptotic expansions of the functions $C(x, a)$ and $S(x, a)$ for large values of x, their Taylor expansions about a point $x = x_1 > 0$ and the order relations (67). Equation (68) then transforms into

$$-D^{-1}g\rho\zeta = 2^{\frac{1}{2}}vtr^{-1}(t+t_1)^{-n}J_1[g(t+t_1)^2 vt/4r^2]\cos[g(t+t_1)^2/4r]. \qquad (69)$$

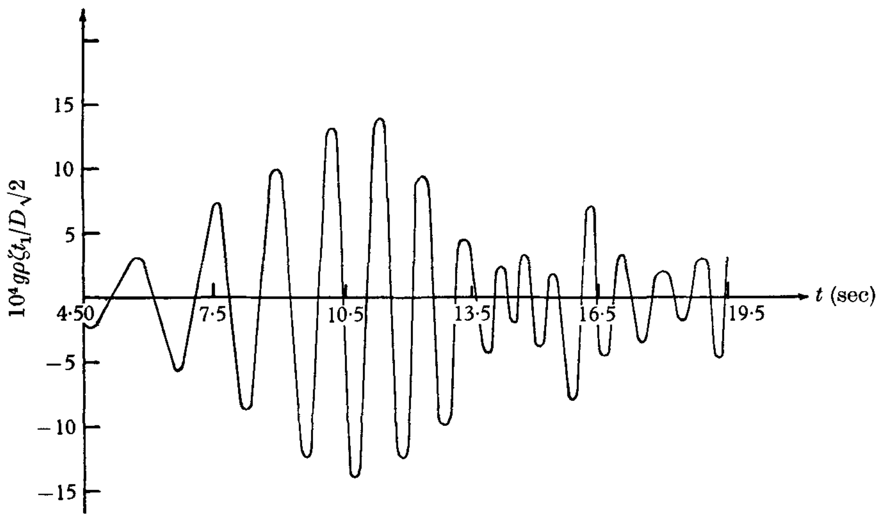

FIGURE 2

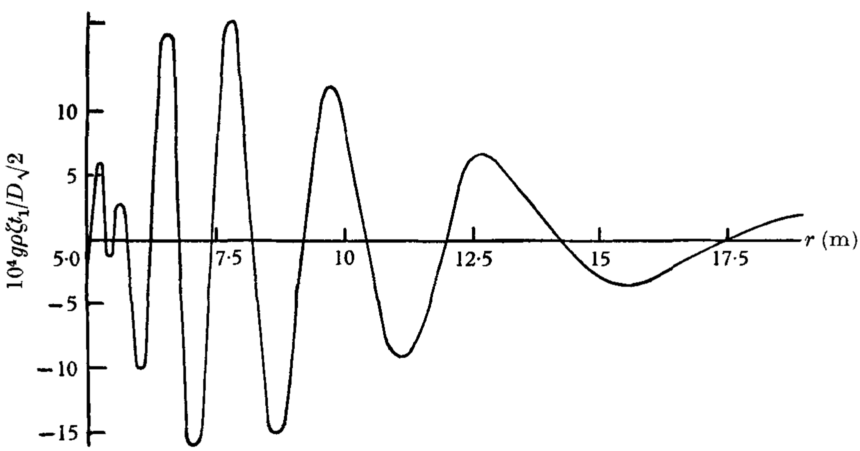

FIGURE 3

For $t_1 = 0{\cdot}5$ sec, $v = 0{\cdot}05$ m/sec, the variation of ζ with t at a distance $r = 10$ m is illustrated in figure 2 and the variation of ζ with r for $t = 9{\cdot}5$ sec is illustrated in figure 3.

(ii) *Case of a weightless fluid.* As in the two-dimensional case, the pressure effects are more marked than the gravity influence in the immediate area of the blast. We therefore proceed to investigate the motion of a weightless fluid both inside the expanding pressure area and in its outer neighbourhood. Letting $g = 0$ in (50) and (51), we get

$$(\phi)_{g=0} = \rho^{-1}\int_0^\infty ke^{kz}J_0(kr)\,dk\int_0^t ds\int_0^{r_0(s)} \alpha f(\alpha,s)\,J_0(k\alpha)\,d\alpha, \tag{70}$$

$$(\zeta)_{g=0} = -\rho^{-1}\lim_{z\to 0-}\int_0^\infty k^2e^{kz}J_0(kr)\,dk\int_0^t (t-s)\,ds\int_0^{r_0(s)} \alpha f(\alpha,s)\,J_0(k\alpha)\,d\alpha. \tag{71}$$

The value of $\partial\phi/\partial t$ given by (70) agrees with that obtained, in other ways, by Rumiantsev (1960, equation (2.2)). We have from (59)

$$(\zeta)_{g=0} = \rho^{-1}r^{-3}\int_0^t (t-s)\,ds\int_0^{r_0(s)} \alpha f(\alpha,s)\,{}_2F_1\left(\tfrac{3}{2},\tfrac{3}{2};1;\frac{\alpha^2}{r^2}\right)d\alpha \quad (r > r_0(t)). \tag{72}$$

A quadratic transformation of the hypergeometric series of (72) leads to the result

$$(\zeta)_{g=0} = (2/\pi\rho)\int_0^t (t-s)\,ds\int_0^{r_0(s)} \alpha(r+\alpha)^{-1}(r-\alpha)^{-2}f(\alpha,s)\,E(K_1)\,d\alpha \quad (r > r_0(t)), \tag{73}$$

where K_1^2 is defined by (57), and $E(K_1)$ is the complete elliptic integral of the second kind.

Particular cases

(1) Let

$$\left.\begin{aligned} f(r,t) &= Dr^mt^p \quad (m > -\tfrac{3}{2},\ m+p+3 > 0),\\ r_0(t) &= vt. \end{aligned}\right\} \tag{74}$$

Then, from (61),

$$\begin{aligned}(\zeta)_{g=0} = {}&\{B(1,\tfrac{1}{2}m+1)/4\rho r^3\}\,Dt^{p+2}(vt)^{m+2}[B(1,\tfrac{1}{2}p+\tfrac{1}{2}m+\tfrac{3}{2})\\ &\times {}_4F_3(\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}m+1,\tfrac{1}{2}p+\tfrac{1}{2}m+\tfrac{3}{2};1,\tfrac{1}{2}m+2,\tfrac{1}{2}p+\tfrac{1}{2}m+\tfrac{5}{2};v^2t^2/r^2)\\ &-B(1,\tfrac{1}{2}p+\tfrac{1}{2}m+2)\,{}_4F_3(\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}m+1,\tfrac{1}{2}p+\tfrac{1}{2}m+2;1,\tfrac{1}{2}m+2,\tfrac{1}{2}p+\tfrac{1}{2}m+3;\ v^2t^2/r^2)],\\ &\qquad (r > vt).\end{aligned} \tag{75}$$

For $p = 0$, the second of the two ${}_4F_3$ functions of (75) becomes $a\,{}_3F_2$ function. For $m = 0$, $p = 0$ the first of the ${}_4F_3$ functions of (75) reduces to a ${}_3F_2$ function while the second becomes a ${}_2F_1$ function.

(2) Let

$$f(r,t) = D, \quad r_0(t) = (vt)^{\frac{1}{2}}, \quad (D,v) = \text{const.} \tag{76}$$

This hypothesis for the pressure distribution may be supposed as approximately correct everywhere except in the vicinity of the shock wave. Equation (71) gives

$$\begin{aligned}-\rho D^{-1}(\zeta)_{g=0} &= v^{\frac{1}{2}}\int_0^\infty kJ_0(kr)\,dk\int_0^t (t-s)\,s^{\frac{1}{2}}\,J_1(kv^{\frac{1}{2}}s^{\frac{1}{2}})\,ds\\ &= 4t^{\frac{3}{2}}v^{-\frac{1}{2}}\int_0^\infty k^{-1}J_0(kr)\,J_3(kv^{\frac{1}{2}}t^{\frac{1}{2}})\,dk.\end{aligned}$$

The k-integral on the right-hand side above is a discontinuous integral of the Weber-Schafheitlin type. Evaluating it, we finally obtain

$$\left.\begin{aligned}\rho D^{-1}(\zeta)_{g=0} &= -\tfrac{4}{3}v^{-\frac{1}{2}}t^{\frac{3}{2}}\,{}_2F_1(\tfrac{3}{2},-\tfrac{1}{2};1;r^2/vt) \quad (r^2 < vt)\\ &= (vt^3/12r^3)\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};4;vt/r^2) \qquad (r^2 > vt).\end{aligned}\right\} \tag{77}$$

It may be shown by using the Gauss value for ${}_2F_1(\alpha,\beta;\gamma;1)$ that the limit $\lim_{r^2 \to vt} (\zeta)_{g=0}$ exists finitely. The surface displacement has thus always an ordinary discontinuity at the boundary $r = (vt)^{\frac{1}{2}}$ of the pressure region.

4. Discussion of the motion

Let us consider the wave elevation as given by (69) for large values of time at a large distance from the pressure distribution (65). As the cosine factor in (69) changes its sign rapidly, its argument may be interpreted as the phase, and its co-factor as the amplitude. We note that the phase is not directly affected by the velocity parameter v. For $t \gg t_1$, the amplitude varies as $-vt^{1-n}r^{-1}J_1(gvt^3/4r^2)$. The time of maximum amplitude at any point is given by $t = (4r^2a_m/gv)^{\frac{1}{3}}$, where a_m is the mth root of the equation

$$3xJ_0(x) = (n+2)\,J_1(x).$$

Therefore the points of maximum amplitude at a distance r move outwards with the corresponding velocities $(27\,gvr/32a_m)^{\frac{1}{3}}$. This must be the group velocity for the predominant wavelength near the maximum; thus the value of this wavelength is $\lambda_m = \pi r/a_m$. At times $t = (4r^2b_m/gv)^{\frac{1}{3}}$, where b_m is the mth root of the equation $J_1(x) = 0$, the amplitude becomes almost zero. The points of minimum amplitude at a distance r move outwards with the corresponding velocities $(27gvr/32b_m)^{\frac{1}{3}}$. The smaller m is, so also are a_m and b_m. Therefore, the outer ring spreads faster than the inner one. Again, when $(gvt^3/4r^2) \gg 1$, the asymptotic expansion of $J_1(gvt^3/4r^2)$ shows that the amplitude is almost of the order of $t^{-(n+\frac{1}{2})}$. In the two-dimensional analogue (28) of the above formula for ζ, the time of maximum amplitude at distance x for $t \gg t_1$ is given by

$$t = (4x^2e_m/gv)^{\frac{1}{3}},$$

where e_m is the mth root of the equation

$$\tan x/x = 3/(n+1).$$

Equation (59) shows that at any instant, the waves die out like r^{-3} at large distances from the seat of the applied pressure. In the two-dimensional problem, as equations (17) and (15) show, the corresponding law of decay is that of the inverse square in x.

Displacement at the origin in a weightless fluid

In (71), let
$$r = 0, \quad f(\alpha, s) = \lambda(s)\,\alpha^{2m} \quad (m = 0, 1, 2, \ldots), \tag{78}$$
and let the orders of the k- and s-integrations be inverted. Now

$$\int_0^{r_0(s)} \alpha f(\alpha,s)\,J_0(k\alpha)\,d\alpha = m!\,\lambda(s)\,r_0^{2m+2}\sum_{n=0}^{m}\frac{(-1)^n\,2^n}{(m-n)!}\,\frac{J_{n+1}(kr_0)}{(kr_0)^{n+1}}.$$

After evaluating the limit of the k-integral of (71) as $z \to 0-$, we find, at $r = 0$,

$$(\zeta)_{g=0} = -\pi\rho^{-1}\int_0^t (t-s)\,\lambda(s)\,\{r_0(s)\}^{2m-1}ds\left\{1+\pi^{-\frac{1}{2}}\sum_{n=1}^{m}\frac{(-1)^n\,m!}{(m-n)!\,\Gamma(n+\frac{1}{2})}\right\}. \tag{79}$$

If h denotes the height of the centre of blast above the liquid, t_1 the time-interval between the blast and the first arrival of the shock at the liquid surface, $R_0(t_1+s)$ the radius of the shock-front in the gas at time t_1+s after the blast,

$$[r_0(s)]^{2m-1} = [\{R_0(t_1+s)\}^2 - h^2]^{m-\frac{1}{2}}$$
$$= O(s^{m-\frac{1}{2}}) \quad \text{for small values of } s.$$

If, therefore, $\lambda(s) = o(s^{-m-\frac{1}{2}})$ for small values of s, the vertical surface displacement at the origin (the point which is directly under the centre of the blast) remains bounded at all times for arbitrary rates of shock propagation in the gas.

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REFERENCES

Erdélyi, A. (Ed.) 1953 *Higher Transcendental Functions*, vol. I–III. New York: McGraw-Hill.

Erdélyi, A. (Ed.) 1954 *Tables of Integral Transforms*, vol. I–II. New York: McGraw-Hill.

Kisler, V. M. 1960 *Prikl. Math. Mek.* **24**, 496–503.

Lamb, H. 1932 *Hydrodynamics*. Cambridge University Press.

Rumiantsev, B. N. 1960 *Prikl. Math. Mek.* **24**, 240–248.

Ryshik, I. M. & Gradstein, I. S. 1957 *Tables of Series, Products and Integrals*. Berlin: Veb Deutscher Verlag der Wissenschaften.

Whittaker, E. T. & Watson, G. N. 1952 *Modern Analysis*. Cambridge University Press.